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The Laplacian spectral radius of trees and maximum vertex degree

Xi-Ying Yuan [a,*], Yue Liu [b], Miaomiao Han [a]

[a] Department of Mathematics, Shanghai University, Shanghai, 200444, China

[b] College of mathematics and computer science, Fuzhou University, Fuzhou, 350108, China

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ABSTRACT

Let $\varDelta(T)$ and $\mu(T)$ denote the maximum degree and the Laplacian spectral radius of a tree T, respectively. In this paper we prove that for two trees T_1 and T_2 on n ($n \geq 21$) vertices, if $\varDelta(T_1) > \varDelta(T_2)$ and $\varDelta(T_1) \geq \lceil \frac{11n}{30} \rceil + 1$, then $\mu(T_1) > \mu(T_2)$, and the bound "$\varDelta(T_1) \geq \lceil \frac{11n}{30} \rceil + 1$" is the best possible. We also prove that for two trees T_1 and T_2 on $2k$ ($k \geq 4$) vertices with perfect matchings, if $\varDelta(T_1) > \varDelta(T_2)$ and $\varDelta(T_1) \geq \lceil \frac{k}{2} \rceil + 2$, then $\mu(T_1) > \mu(T_2)$.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph. Denote by $N(v)$ the set of all the neighbors of a vertex v in G, and by $d(v)$ the degree of v. Denote by $m(v)$ the average of the degrees of the vertices adjacent to v, i.e., $m(v) = \frac{\sum_{u \in N(v)} d(u)}{d(v)}$.

Let $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix $L(G)$ of G is defined by $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G. In this paper, the characteristic polynomial $\det(xI - L(G))$ is denoted by $\Phi(L(G); x)$ (or simply $\Phi(L(G))$). We always use $\mu_i(B)$ to denote the ith eigenvalue of a real symmetric matrix B (of order n), and the inequalities $\mu_1(B) \geq \mu_2(B) \geq \cdots \geq \mu_n(B)$ hold. Write $\mu_i(G) = \mu_i(L(G))$ for short. The eigenvalue $\mu_1(G)$ is called the Laplacian spectral radius of G, denoted by $\mu(G)$. We may cite some recent papers (see [5,6,12,13,20]) on the ordering of trees based on the Laplacian spectral radius and Laplacian coeffcients.

Denote by $\varDelta(G)$ for the maximum degree of G. Let $\mathcal{T}(n)$ be the set of trees on n vertices. It is proved in [18] that the Laplacian spectral radius of a tree T in $\mathcal{T}(n)$ ($n \geq 6$) strictly increases with its maximum degree when $\varDelta(T) \geq \lceil \frac{n}{2} \rceil + 1$. In this paper, we prove that the bound "$\varDelta(T) \geq \lceil \frac{n}{2} \rceil + 1$" can be improved as "$\varDelta(T) \geq \lceil \frac{11n}{30} \rceil + 1$", and we point out that the bound "$\varDelta(T) \geq \lceil \frac{11n}{30} \rceil + 1$" is the best possible.

Let $\mathbf{T}(2k)$ be the set of trees on $2k$ vertices with perfect matchings, and $\mathbf{T}(2k, \varDelta) = \{T \in \mathbf{T}(2k) \mid \varDelta(T) = \varDelta\}$. In this paper, we determine the unique tree that takes the largest values of $\mu(T)$ of the trees T in $\mathbf{T}(2k, \varDelta)$, when $\varDelta \geq \lceil \frac{k}{2} \rceil + 1$ (see Theorem 3.1). For two trees T_1 and T_2 in $\mathbf{T}(2k)$ ($k \geq 4$) we prove that if $\varDelta(T_1) > \varDelta(T_2)$ and $\varDelta(T_1) \geq \lceil \frac{k}{2} \rceil + 2$, then $\mu(T_1) > \mu(T_2)$.

2. A relation between $\mu(T)$ and $\varDelta(T)$ of trees in $\mathcal{T}(n)$

Let $\mathcal{T}(n)$ be the set of trees on n vertices, and $\mathcal{T}(n, \varDelta) = \{T \in \mathcal{T}(n) \mid \varDelta(T) = \varDelta\}$. Denote by T^* the tree with the largest Laplacian spectral radius among all the trees in $\mathcal{T}(n, \varDelta)$. The following characterization for T^* is due to Yu (see [19]).

* Corresponding author.
E-mail address: xiyingyuan2007@hotmail.com (X.-Y. Yuan).

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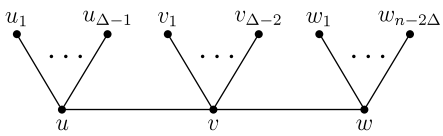

Fig. 1. The tree $F(n, \Delta)$.

Lemma 2.1 ([19]). *Let T^* be a tree with the largest Laplacian spectral radius among all the trees in $\mathcal{T}(n, \Delta)$. Then*

(1) *it is a rooted tree with c as a root;*
(2) *its height with respect to c is equal to h (for some h);*
(3) *each pendant vertex of T^* is at distance $h - 1$ or h from c;*
(4) *each vertex, except possible one, is of degree 1 or Δ;*
(5) *the vertex of degree d $(1 < d < \Delta)$, if any, is at distance $h - 1$ from c.*

It is easy to see that the vertex c of T^* has maximum degree. The height $h = 1$ if and only if $\Delta = n - 1$, and $h = 2$ if and only if $\sqrt{n-1} \le \Delta \le n - 2$. When the height of T^* is at most 2, then T^* is determined by Lemma 2.1 up to isomorphism. When $\lceil \frac{n+6}{3} \rceil \le \Delta \le \lceil \frac{n}{2} \rceil - 1$ $(n \ge 18)$ the tree T^* has height 2, so we may obtain the following result by Lemma 2.1.

Corollary 2.1. *Let T be a tree in $\mathcal{T}(n, \Delta)$, and $F(n, \Delta)$ be the tree as shown in Fig. 1. When $\lceil \frac{n+1}{3} \rceil \le \Delta \le \lceil \frac{n}{2} \rceil - 1$ $(n \ge 18)$, we have $\mu(T) \le \mu(F(n, \Delta))$, and equality holds if and only if $T \cong F(n, \Delta)$.*

The tree $S(n, \Delta)$ on n vertices is called a double star graph, which is obtained by joining the center of $K_{1,\Delta-1}$ to that of $K_{1,n-\Delta-1}$ by an edge, where $\Delta \ge \lceil \frac{n}{2} \rceil$. Let $G_1 u : v G_2$ be the graph obtained by joining the vertex u of the graph G_1 to the vertex v of the graph G_2 by an edge, where G_1 and G_2 are disjoint. Then we have $F(n, \Delta) \cong S(2\Delta - 1, \Delta)v : wK_{1,n-2\Delta}$, where $F(n, \Delta)$, v and w are shown in Fig. 1.

If $v \in V(G)$, let $L_v(G)$ be the principal sub-matrix of $L(G)$ obtained by deleting the row and column corresponding to the vertex v. The following result is due to Guo [9].

Lemma 2.2 ([9]). *Let $G \cong G_1 u : v G_2$. Then*

$$\Phi(L(G)) = \Phi(L(G_1))\Phi(L(G_2)) - \Phi(L(G_1))\Phi(L_v(G_2)) - \Phi(L(G_2))\Phi(L_u(G_1)).$$

By using Lemma 2.2 recursively, we may give the characteristic polynomial of matrix $L(F(n, \Delta))$. Let

$$\begin{aligned} f(x) = {} & x^5 - (n+4)x^4 + (2\Delta n + 2n - 3\Delta^2 + 2\Delta + 7)x^3 - (\Delta^2 n + 2\Delta n + 3n - 2\Delta^3 - 2\Delta^2 + 2\Delta + 6)x^2 \\ & + (3\Delta n + n - 5\Delta^2 + 4\Delta + 2)x - n. \end{aligned} \tag{2.1}$$

Lemma 2.3. *Let $F(n, \Delta)$ be the graph as shown in Fig. 1, and the function $f(x)$ be defined in (2.1). We have $\Phi(L(F(n, \Delta)); x) = x(x-1)^{n-6} f(x)$.*

Proof. First we give the characteristic polynomial of matrix $L(S(2\Delta - 1, \Delta))$ by using Lemma 2.2 (we may see [7] or [16] for the characteristic polynomial of matrix $L(S(2\Delta - 1, \Delta))$. As we all know that $\Phi(L(K_{1,\Delta}); x) = x(x - \Delta - 1)(x-1)^{\Delta-1}$, and $\Phi(L_u(K_{1,\Delta}); x) = (x-1)^{\Delta}$, where u is the center of the star $K_{1,\Delta}$. Now take the non-pendant edge of $S(2\Delta - 1, \Delta)$ as the edge uv of G in Lemma 2.2. We have

$$\begin{aligned} \Phi(L(S(2\Delta-1, \Delta)); x) &= x^2(x-\Delta)(x-\Delta+1)(x-1)^{2\Delta-5} - x(x-\Delta)(x-1)^{2\Delta-4} - x(x-\Delta+1)(x-1)^{2\Delta-4} \\ &= x(x-1)^{2\Delta-5}[x^3 - (2\Delta+1)x^2 + (\Delta^2+\Delta+1)x - (2\Delta-1)]. \end{aligned}$$

It is easy to check that $\Phi(L_v(S(2\Delta - 1, \Delta)); x) = (x-1)^{2\Delta-4}[x^2 - (\Delta+1)x + 1]$, where v is the vertex of the double star graph $S(2\Delta - 1, \Delta)$ with degree $\Delta - 1$.

Now we give the characteristic polynomial of matrix $L(F(n, \Delta))$. Take the edge vw of $F(n, \Delta)$ as the edge uv of G in Lemma 2.2.

$$\begin{aligned} \Phi(L(F(n, \Delta)); x) &= x^2(x-1)^{n-6}(x-n+2\Delta-1)[x^3 - (2\Delta+1)x^2 + (\Delta^2+\Delta+1)x - (2\Delta-1)] \\ &\quad - x(x-1)^{n-5}[x^3 - (2\Delta+1)x^2 + (\Delta^2+\Delta+1)x - (2\Delta-1)] \\ &\quad - x(x-1)^{n-5}(x-n+2\Delta-1)[x^2 - (\Delta+1)x + 1] \\ &= x(x-1)^{n-6} f(x). \quad \square \end{aligned}$$

Lemma 2.4 ([7]). *Let G be a connected graph on n vertices having at least one edge, then $\mu(G) \ge \Delta(G) + 1$, with equality if and only if $\Delta(G) = n - 1$.*

The following inequalities are known as Cauchy's inequalities.

Lemma 2.5. *Let H be a Hermitian matrix of order n and H' be a principal submatrix of H of order m. Then the inequalities $\mu_{n-m+i}(H) \le \mu_i(H') \le \mu_i(H)$ $(i = 1, 2, \ldots, m)$ hold.*

Lemma 2.6. *When $\lceil \frac{11n}{30} \rceil \le \Delta \le \lceil \frac{n}{2} \rceil - 1$ $(n \ge 10)$, we have $\mu(F(n, \Delta)) < \Delta + 2$.*

Proof. First we prove that $\mu_2(F(n, \Delta)) \le \Delta + 1$. Using Lemma 2.5 twice, we have

$$\begin{aligned} \mu_2(F(n, \Delta)) &\le \mu_1(L_v(F(n, \Delta))) \\ &\le \max\{\mu_1(K_{1,\Delta}), \mu_1(K_{1,n-2\Delta+1})\} \\ &= \Delta + 1, \end{aligned}$$

where v is the vertex of $F(n, \Delta)$ as shown in Fig. 1. So $\mu_2(F(n, \Delta)) \le \Delta + 1$ holds.

From Lemma 2.3 we have

$$\Phi(L(F(n, \Delta)); x) = x(x-1)^{n-6} f(x). \tag{2.2}$$

From (2.2) we know that $\mu(F(n, \Delta))$ is the largest root of the equation $f(x) = 0$. And when $\lceil \frac{11n}{30} \rceil \le \Delta \le \lceil \frac{n}{2} \rceil - 1$, it is easy to verify that

$$f(\Delta + 1) = -[\Delta^2(3\Delta - n - 1) + 2(n - 2\Delta)] < 0, \tag{2.3}$$

$$f(\Delta + 2) = 4 + (30\Delta - 11n) + (14\Delta^2 - 5\Delta n) > 0. \tag{2.4}$$

From (2.3) and (2.4) we know that there exists a root of the equation $f(x) = 0$ in the interval $(\Delta + 1, \Delta + 2)$. Note that $\mu_2(F(n, \Delta)) \le \Delta + 1$, so we have $\mu(F(n, \Delta)) < \Delta + 2$. □

Lemma 2.7 ([14]). *Let G be a graph on n vertices. Then $\mu(G) \le \max\{d(v) + m(v) \mid v \in V(G)\}$.*

Lemma 2.8. *Let T be a tree on n $(n \ge 21)$ vertices with $\Delta(T) \le \lceil \frac{11n}{30} \rceil - 1$. Then $\mu(T) < \lceil \frac{11n}{30} \rceil + 2$.*

Proof. Let v be a vertex of tree T and write $d(v) = s$. Now we distinguish the following two cases depending on the value of s.

Case 1. $s \le 2$.

$$d(v) + m(v) \le 2 + \Delta(T) \le 2 + \left\lceil \frac{11n}{30} \right\rceil - 1 < \left\lceil \frac{11n}{30} \right\rceil + 2. \tag{2.5}$$

Case 2. $3 \le s \le \lceil \frac{11n}{30} \rceil - 1$.

Denote by $|E(T)|$ the number of edges of the tree T. Since tree T is a bipartite graph and all neighbors of v belong to exactly one partition, we have

$$m(v) = \frac{1}{s} \sum_{u \in N(v)} d(u) \le \frac{1}{s} |E(T)| = \frac{n-1}{s}.$$

Hence

$$d(v) + m(v) \le s + \frac{n-1}{s}.$$

Let $g(s) = s + \frac{n-1}{s}$. Then $g''(s) = \frac{2(n-1)}{s^3} > 0$, and so $g(s)$ is convex when $s > 0$ and $n \ge 2$. Hence when $3 \le s \le \lceil \frac{11n}{30} \rceil - 1$, we have

$$g(s) \le \max\left\{ g(3), g\left(\left\lceil \frac{11n}{30} \right\rceil - 1 \right) \right\} = \max\left\{ 3 + \frac{n-1}{3}, \left\lceil \frac{11n}{30} \right\rceil - 1 + \frac{n-1}{\lceil \frac{11n}{30} \rceil - 1} \right\}.$$

When $n \ge 21$, it is easy to check that $3 + \frac{n-1}{3} < \lceil \frac{11n}{30} \rceil + 2$, and $\frac{n-1}{\lceil \frac{11n}{30} \rceil - 1} < 3$. So $g(s) < \lceil \frac{11n}{30} \rceil + 2$.

Finally, we have

$$d(v) + m(v) < \left\lceil \frac{11n}{30} \right\rceil + 2. \tag{2.6}$$

Combining (2.5) and (2.6) and Lemma 2.7 we have $\mu(T) < \lceil \frac{11n}{30} \rceil + 2$. □

The following result was proved in [18] (see Theorem 2.1 and Lemma 3.1 of [18]).

Lemma 2.9 ([18]). *Let T be a tree in $\mathcal{T}(n, \Delta)$. When $\Delta \ge \lceil \frac{n}{2} \rceil$ $(n \ge 6)$, we have $\mu(T) \le \mu(S(n, \Delta)) < \Delta + 2$.*

Now we are ready to give the main result of this section.

Theorem 2.1. *Let $T_1, T_2 \in \mathcal{T}(n)$ $(n \geq 21)$. If $\Delta(T_1) > \Delta(T_2)$ and $\Delta(T_1) \geq \lceil \frac{11n}{30} \rceil + 1$, then $\mu(T_1) > \mu(T_2)$.*

Proof. (1) If $\Delta(T_2) \geq \lceil \frac{n}{2} \rceil$, from Lemmas 2.4 and 2.9, we have

$$\mu(T_1) \geq \Delta(T_1) + 1 \geq \Delta(T_2) + 2 > \mu(S(n, \Delta(T_2))) \geq \mu(T_2).$$

(2) If $\lceil \frac{11n}{30} \rceil \leq \Delta(T_2) \leq \lceil \frac{n}{2} \rceil - 1$, from Lemma 2.4, Corollary 2.1, and Lemma 2.6 we have

$$\mu(T_1) \geq \Delta(T_1) + 1 \geq \Delta(T_2) + 2 > \mu(F(n, \Delta(T_2))) \geq \mu(T_2).$$

(3) If $\Delta(T_2) \leq \lceil \frac{11n}{30} \rceil - 1$, from Lemmas 2.4 and 2.8 we have

$$\mu(T_1) \geq \Delta(T_1) + 1 \geq \left\lceil \frac{11n}{30} \right\rceil + 2 > \mu(T_2).$$

The proof is completed. $\square$

Denote by $B(n, \Delta)$ $(2 \leq \Delta \leq n-1)$ the tree on n vertices, which is obtained by joining one pendant vertex of $K_{1,\Delta}$ to one pendant vertex of the path $P_{n-\Delta-1}$ by an edge. We may point out that the tree $B(n, \Delta)$ has simultaneously maximal all Laplacian coefficients among trees with n vertices and maximal vertex degree Δ (see [12]).

Remark 2.1. Theorem 2.1 indicates that the Laplacian spectral radius of a tree T in $\mathcal{T}(n)$ $(n \geq 21)$ strictly increases with its maximum degree when $\Delta(T) \geq \lceil \frac{11n}{30} \rceil + 1$. The bound "$\Delta(T) \geq \lceil \frac{11n}{30} \rceil + 1$" is the best possible. To see this, let $T_1 = B(90, 33)$ and $T_2 = F(90, 32)$ (tree $F(n, \Delta)$ is shown in Fig. 1). By using the software *Mathematica* we have $\mu(T_1) = 34.0009$ (up to four decimal places) and $\mu(T_2) = 34.0066$. We have $\Delta(T_1) > \Delta(T_2)$ while $\mu(T_1) < \mu(T_2)$.

3. The tree with the largest Laplacian spectral radius in $\mathbf{T}(2k, \Delta)$

Let $\mathbf{T}(2k, \Delta) = \{T \in \mathbf{T}(2k) \mid \Delta(T) = \Delta\}$. In this section we assume that $\Delta \geq \lceil \frac{k}{2} \rceil + 1$ $(k \geq 4)$, then each tree in $\mathbf{T}(2k, \Delta)$ contains exactly one vertex with degree Δ. Let T^* be a tree from $\mathbf{T}(2k, \Delta)$ whose Laplacian spectral radius attains the maximum value. In this section we will characterize the tree T^*. First we formulate some basic tools.

Let $P = v_0 v_1 \cdots v_k$ $(k \geq 1)$ be a path of a tree T. If $d_T(v_0) \geq 3$, $d_T(v_k) \geq 3$ and $d_T(v_i) = 2$ $(0 < i < k)$, then P is called an *internal path* of T (see [10]).

Lemma 3.1 ([17]). *Let uv be an edge on an internal path of tree T. Denote by $T^\sharp$ the tree obtained from T by contracting the edge uv, i.e., identifying the vertices u and v in T. Then $\mu(T^\sharp) > \mu(T)$.*

Lemma 3.2 ([8]). *Let G be a connected bipartite graph, and G' be a subgraph of G. Then $\mu(G) \geq \mu(G')$, and equality holds if and only if $G \cong G'$.*

Lemma 3.3 ([15]). *$L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ have the same spectrum if and only if G is a bipartite graph.*

The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian of graph G. See [2–4] for some recent results about the theory of signless Laplacians of graphs. By Lemma 3.3, for any tree T, its Laplacian spectral radius $\mu(T)$ is equal to the largest eigenvalue of $Q(T)$ $(\mu_1(Q(T)))$. Since $Q(T)$ is a nonnegative irreducible positive semidefinite symmetric matrix, there is a positive unit eigenvector X belonging to $\mu(T)$. In the following proof, if v is a vertex of a graph G, then x_v denotes the element of X corresponding to v.

Lemma 3.4 ([11]). *Let u, v be two vertices of a connected graph G. Suppose $v_1, v_2, \ldots, v_s (1 \leq s \leq d(v))$ are some vertices of $N_G(v) \setminus (N_G(u) \bigcup \{u\})$. Let G' be the graph obtained from G by deleting the edges $vv_1, vv_2, \ldots, vv_s$ and adding the edges $uv_1, uv_2, \ldots, uv_s$. Let X be the unit eigenvector of $Q(G)$ corresponding to $\mu_1(Q(G))$. If $x_u \geq x_v$, then $\mu_1(Q(G')) > \mu_1(Q(G))$.*

Lemma 3.5 ([19]). *Let st and uv be two edges of a tree T, and $sv, tu \notin E(T)$. Let T' be the tree obtained from T by deleting the edges st and uv, and adding edges sv and tu. Let X be the eigenvector of $Q(T)$ corresponding to $\mu(T)$. Then the following holds: If $(x_s - x_u)(x_v - x_t) \geq 0$, then $\mu(T') \geq \mu(T)$, and the equality holds if and only if $x_s = x_u$ and $x_v = x_t$.*

In the following proofs, we always suppose that w is the vertex of T^* with degree Δ, and X is the unit eigenvector of $Q(T^*)$ corresponding to $\mu(T^*)$. If a tree T has a perfect matching, then the matching is unique, denoted by $M(T)$ in this paper.

Lemma 3.6. *There exists at most one vertex, say u, of T^* with degree $2 < d(u) < \Delta$.*

Proof. Suppose to the contrary that there exist two vertices, say u, v, such that $2 < d(u), d(v) < \Delta$. Without loss of generality suppose that $x_u \geq x_v$. The assumption $d(v) > 2$ ensures the existence of a vertex in $N(v)$, say t, such that t does not lie on the (unique) path between v and u, and $tv \notin M(T^*)$. Let

$$T' = T^* - vt + ut,$$

then T' is also in $\mathbf{T}(2k, \Delta)$. Thus $\mu(T') > \mu(T^*)$ follows from Lemma 3.4. This contradicts to the definition of T^*. □

Lemma 3.7. *If T^* has a vertex, say u, with degree $2 < d(u) < \Delta$, then $uw \in E(T^*)$.*

Proof. Denote by $uu_1 \cdots u_s$ $(=w)$ the (unique) path between u and w. If $u_1 = w$, we are done. So $u_1 \neq w$, and the edge uu_1 lies on some internal path of T^*. We distinguish the following two cases.

Case 1. $uu_1 \in M(T^*)$.

We may contract the edge uu_1, and then attach a pendant edge to the vertex u, the resulting tree is denoted by T'. It is easy to see that $T' \in \mathbf{T}(2k, \Delta)$. Then $\mu(T') > \mu(T^*)$ follows Lemmas 3.1 and 3.2. This contradicts to the definition of T^*.

Case 2. $uu_1 \notin M(T^*)$.

Suppose $uu' \in M(T^*)$, then $u' \neq w$. Contract the edge uu_1, and then attach a pendant edge to the vertex u', the resulting tree is denoted by T''. Then $T'' \in \mathbf{T}(2k, \Delta)$. So we have $\mu(T'') > \mu(T^*)$, which follows Lemmas 3.1 and 3.2. This contradicts to the definition of T^*.

The proof is completed. □

Lemma 3.8. *If T^* has a vertex, say u, with degree $2 < d(u) < \Delta$, then $uw \notin M(T^*)$.*

Proof. Suppose to the contrary that $uw \in M(T^*)$. Let w_1 be a neighbor of w different from u, and w_1 is saturated by w_2 $(w_2 \neq w)$. And let u_1 be a neighbor of u different from w, and u_1 is saturated by u_2 $(u_2 \neq u)$. Then we may apply the following transformation to T^*.

Set

$$T' = \begin{cases} T^* - w_1w_2 + uw_2, & \text{if } x_u \geq x_{w_1}; \\ T^* - u_1u_2 + w_2u_2, & \text{if } x_{w_2} \geq x_{u_1}; \\ T^* - \{w_1w_2, uu_1\} + \{w_1u_1, uw_2\}, & \text{if } x_{w_1} > x_u \text{ and } x_{u_1} > x_{w_2}. \end{cases}$$

Then $\mu(T') > \mu(T^*)$ follows from Lemmas 3.4 and 3.5, while T' is also in $\mathbf{T}(2k, \Delta)$. This contradicts to the definition of T^*. □

Lemma 3.9 (*[3]*)**.** *Let $G(k; \ell)$ be the graph obtained from a non-trivial connected graph G by attaching pendant paths of lengths k and ℓ at some vertex v. If $k \geq \ell \geq 1$, then $\mu_1(Q(G(k; \ell))) > \mu_1(Q(G(k+1; \ell-1)))$.*

Lemma 3.10. *Each path attached at the vertex w of T^* has length at most 2.*

Proof. First we will prove that each path attached at the vertex w of T^* has length at most 3.

Suppose to the contrary that $w_1w_2w_3w_4 \ldots w_sw$ is one path attached at w, where w_1 is a pendant vertex, and $s \geq 4$. Then by Lemma 3.9 it follows that the tree $T' = T^* - w_2w_1 + w_4w_1$ has a larger Laplacian spectral radius. While T' is also in $\mathbf{T}(2k, \Delta)$, and this contradicts to the definition of T^*.

Now we will prove that each path attached at the vertex w of T^* has length at most 2.

Suppose to the contrary that $ww_{13}w_{12}w_{11}$ is a path of length 3 attached at w. Since T^* has perfect matching, $ww_{13}w_{12}w_{11}$ is the unique path with odd length attached at w. And noting the hypothesis $\Delta \geq 3$, we may suppose $ww_{22}w_{21}$ is a path of length 2 attached at w. Then we may apply the following transformations to T^*.

Set

$$T' = \begin{cases} T^* - w_{13}w_{12} + w_{22}w_{12}, & \text{if } x_{w_{22}} \geq x_{w_{13}}; \\ T^* - w_{22}w_{21} + w_{13}w_{21}, & \text{if } x_{w_{13}} > x_{w_{22}}. \end{cases}$$

Then $\mu(T') > \mu(T^*)$ from Lemma 3.4, while T' is also in $\mathbf{T}(2k, \Delta)$. This contradicts to the definition of T^*. So each path attached at the vertex w of T^* has length at most 2. □

By the similar arguments of Lemma 3.10, we obtain the following results.

Lemma 3.11. *If T^* has a vertex, say u, with degree $2 < d(u) < \Delta$, then each path attached at the vertex u of T^* has length at most 2.*

Combining Lemmas 3.6–3.11, we obtain the main results of this section.

Theorem 3.1. *Let T be a tree in $\mathbf{T}(2k, \Delta)$, and $T^*(2k, \Delta)$ be the tree as shown in Fig. 2. When $\Delta \geq \lceil \frac{k}{2} \rceil + 1$ $(k \geq 4)$, we have $\mu(T) \leq \mu(T^*(2k, \Delta))$, and equality holds if and only if $T \cong T^*(2k, \Delta)$.*

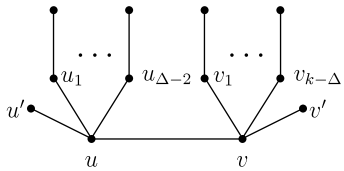

Fig. 2. The tree $T^*(2k, \Delta)$.

4. A relation between $\mu(T)$ and $\Delta(T)$ of a tree T in $\mathbf{T}(2k)$

In this section, we will prove that for two trees T_1 and T_2 in $\mathbf{T}(2k)$ $(k \geq 4)$, if $\Delta(T_1) > \Delta(T_2)$ and $\Delta(T_1) \geq \lceil \frac{k}{2} \rceil + 2$, then $\mu(T_1) > \mu(T_2)$.

Lemma 4.1 ([1]). *Let $G' = G + e$ be the graph obtained from G by inserting a new edge e. Then Laplacian eigenvalues of G interlace the Laplacian eigenvalues of G', i.e.,*

$$\mu_1(G') \geq \mu_1(G) \geq \mu_2(G') \geq \cdots \geq \mu_n(G') \geq \mu_n(G).$$

Lemma 4.2. *Let $T^*(2k, \Delta)$ be the tree as shown in Fig. 2. Then $\mu(T^*(2k, \Delta)) < \Delta + 2$, where $\lceil \frac{k}{2} \rceil + 1 \leq \Delta \leq k$, $k \geq 4$.*

Proof. First we prove that $\mu_2(T^*(2k, \Delta)) < \Delta + 1$. Denote by $\widehat{K_{1,n-1}}$ the tree (on $2n$ vertices) obtained from the star $K_{1,n-1}$ by attaching an edge at each vertex. Let uv be the edge of $T^*(2k, \Delta)$ (see Fig. 2). Then

$$T^*(2k, \Delta) - uv = \widehat{K_{1,k-\Delta}} \bigcup \widehat{K_{1,\Delta-2}}.$$

And

$$\mu(T^*(2k, \Delta) - uv) = \max\left\{\mu(\widehat{K_{1,k-\Delta}}), \mu(\widehat{K_{1,\Delta-2}})\right\}.$$

It is easy to check that $\max\left\{d(v) + m(v) \mid v \in V(\widehat{K_{1,k-\Delta}})\right\} < \Delta + 1$, and $\max\left\{d(v) + m(v) \mid v \in V(\widehat{K_{1,\Delta-2}})\right\} < \Delta + 1$. Hence from Lemmas 2.7 and 4.1 we have $\mu_2(T^*(2k, \Delta)) \leq \mu(T^*(2k, \Delta) - uv) < \Delta + 1$.

Let

$$\begin{aligned} h(x) = {} & x^6 - (k+8)x^5 + (k\Delta - \Delta^2 + 2\Delta + 5k + 23)x^4 - (4k\Delta - 4\Delta^2 + 8\Delta + 9k + 28)x^3 \\ & + (5k\Delta - 5\Delta^2 + 10\Delta + 7k + 13)x^2 - (2k\Delta - 2\Delta^2 + 4\Delta + 3k + 2)x + k. \end{aligned}$$

Using the similar method of Lemma 2.3, and taking the edge uv of $T^*(2k, \Delta)$ as the edge uv of G in Lemma 2.2, we have

$$\Phi(T^*(2k, \Delta); x) = x(x-2)(x^2 - 3x + 1)^{k-4} h(x).$$

Then $\mu(T^*(2k, \Delta))$ is the largest root of the equation $h(x) = 0$. Noting the fact that $\Delta \geq 3$, we have

$$\begin{aligned} h(\Delta + 1) &= -[\Delta^2(\Delta^2 - 2\Delta - 2) + k\Delta + 2\Delta + 1] \\ &< 0, \end{aligned}$$

and combining the fact that $2\Delta > k + 2$, we have

$$\begin{aligned} h(\Delta + 2) &= 2\Delta^5 - k\Delta^4 + 6\Delta^4 - 4k\Delta^3 + 4\Delta^3 - 5k\Delta^2 + \Delta^2 - 3k\Delta + 2\Delta - k \\ &> 8\Delta^4 - 4k\Delta^3 + 4\Delta^3 - 5k\Delta^2 + \Delta^2 - 3k\Delta + 2\Delta - k \\ &> 12\Delta^3 - 5k\Delta^2 + \Delta^2 - 3k\Delta + 2\Delta - k \\ &> 12\Delta^2 - 3k\Delta + 2\Delta - k \\ &= 3\Delta(4\Delta - k) + (2\Delta - k) \\ &> 0. \end{aligned}$$

Then there exists a root of the equation $h(x) = 0$ in the interval $(\Delta + 1, \Delta + 2)$. Note that $\mu_2(T^*(2k, \Delta)) < \Delta + 1$, so we know $\mu(T^*(2k, \Delta)) < \Delta + 2$. Hence the desired result holds. □

Lemma 4.3. *Let T be a tree in $\mathbf{T}(2k)$ with $\Delta(T) \leq \lceil \frac{k}{2} \rceil$. Then $\mu(T) < \lceil \frac{k}{2} \rceil + 3$.*

Proof. Let v be a vertex of T, and write $d(v) = s$ for short. We distinguish the following two cases.

Case 1. $s = 1$.

We have

$$d(v) + m(v) = 1 + m(v) \leq 1 + \left\lceil \frac{k}{2} \right\rceil < \left\lceil \frac{k}{2} \right\rceil + 3. \tag{3.6}$$

Case 2. $2 \leq s \leq \lceil \frac{k}{2} \rceil$.

Let

$$N(v) = \{v_1, v_2, \ldots, v_s\},$$
$$A(v_i) = N(v_i) \setminus \{v\}, \quad i = 1, 2, \ldots, s.$$

Then $A(v_1), A(v_2), \ldots, A(v_s)$ are pairwise disjoint since T is a tree. And note the fact that T has a perfect matching, then

$$\left| \bigcup_{i=1}^{s} A(v_i) \right| \leq \frac{2k - (s+1) - (s-1)}{2} + s - 1 = k - 1.$$

So

$$\sum_{i=1}^{s} |A(v_i)| = \left| \bigcup_{i=1}^{s} A(v_i) \right| \leq k - 1.$$

Then we have

$$\sum_{u \in N(v)} d(u) \leq s + k - 1.$$

Hence

$$d(v) + m(v) \leq s + \frac{s + k - 1}{s} = s + 1 + \frac{k-1}{s}.$$

Let $f(s) = s + 1 + \frac{k-1}{s}$. Then $f(s) = s + 1 + \frac{k-1}{s}$ is convex when $s > 0$. Hence when $2 \leq s \leq \lceil \frac{k}{2} \rceil$, we have

$$f(s) \leq \max \left\{ f(2), f\left(\left\lceil \frac{k}{2} \right\rceil \right) \right\} < \left\lceil \frac{k}{2} \right\rceil + 3.$$

So when $2 \leq s \leq \lceil \frac{k}{2} \rceil$, we have

$$d(v) + m(v) < \left\lceil \frac{k}{2} \right\rceil + 3. \tag{3.7}$$

Combining (3.6), (3.7) and Lemma 2.7 we have $\mu(T) < \lceil \frac{k}{2} \rceil + 3$. □

Now we are ready to obtain our main result of this section.

Theorem 4.1. *Let T_1, T_2 be trees in $\mathbf{T}(2k)$ $(k \geq 4)$. If $\Delta(T_1) > \Delta(T_2)$ and $\Delta(T_1) \geq \lceil \frac{k}{2} \rceil + 2$, then $\mu(T_1) > \mu(T_2)$.*

Proof. We distinguish the following two cases.

Case 1. $\Delta(T_2) \geq \lceil \frac{k}{2} \rceil + 1$.

From Lemma 2.4, 4.2 and Theorem 3.1, we have

$$\mu(T_1) \geq \Delta(T_1) + 1 \geq \Delta(T_2) + 2 > \mu(T^*(2k, \Delta(T_2))) \geq \mu(T_2),$$

where $T^*(2k, \Delta(T_2))$ is the tree defined in Theorem 3.1.

Case 2. $\Delta(T_2) \leq \lceil \frac{k}{2} \rceil$.

From Lemmas 2.4 and 4.3 we have

$$\mu(T_1) \geq \Delta(T_1) + 1 \geq \left\lceil \frac{k}{2} \right\rceil + 3 > \mu(T_2).$$

The proof is completed. □

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Further reading

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